Exam Lie Groups in Physics

Date	January 31, 2019
Room	EA 5159.0114
Time	9:00 - 12:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	9	2a)	8	3a)	9	4a)	10
1b)	9	2b)	10	3b)	9	4b)	10
1c)	9					4c)	7

$$\text{Result } = \frac{\sum \text{points}}{10} + 1$$

Problem 1
Consider the group $O(3)$ of real orthogonal 3×3 matrices.

(a) Show the isomorphism $O(3)/SO(3) \cong \mathsf{Z}_2$.

(b) Show that the symmetric and antisymmetric tensors $x_i y_j \pm x_j y_i$ do not mix under $O(3)$ transformations.

(c) Count the dimensions of the representations of $O(3)$ under which the symmetric and antisymmetric tensors $x_i y_j \pm x_j y_i$ transform and argue whether they are irreducible or not.

Problem 2
Consider the four-dimensional representation of the generators of the Lorentz group:

$$(M^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha} g^{\nu}{}_{\beta} - g^{\nu\alpha} g^{\mu}{}_{\beta})$$

(a) Write down the matrices for the following two cases: $\mu = 2, \nu = 0$ and $\mu = 2, \nu = 3$.

(b) Derive the matrix expression for $(\exp(i\chi M^{20}))^{\alpha}{}_{\beta}$ using $(M^{20})^{\alpha}{}_{\beta}$ obtained in part (a) and explain which Lorentz transformation it corresponds to.

Problem 3
Consider the Cartan matrix $A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$, where the α_i denote the simple roots, for the complex rank-2 Lie algebra $\widetilde{g}_2$:

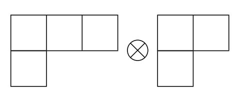
$$\boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

(a) Use this matrix to deduce the angle between and relative lengths of the two simple roots and draw the corresponding Dynkin diagram. Recall that in a Dynkin diagram a single/double/triple line connecting two circles denotes an angle of 120°/135°/150° and an arrow points to the shorter root.

(b) Obtain the root diagram of $\widetilde{g}_2$ by using Weyl reflections and deduce the dimension of the Lie algebra $\widetilde{g}_2$.

Problem 4
Consider the Lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(2)$ and $su(3)$. Indicate the complex conjugate and inequivalent irreps whenever appropriate.

(c) Consider for $su(2)$ the complex conjugate of the above direct product in terms of Young tableaux, decompose it into irreps and compare the answer to the one obtained in part (b). Draw a general conclusion about the (in)equivalence of the irreps of $su(2)$ and their complex conjugates.